

THE TOPOLOGICAL STRUCTURE OF CONTACT AND SYMPLECTIC QUOTIENTS

EUGENE LERMAN AND CHRISTOPHER WILLETT

ABSTRACT. We show that if a Lie group acts properly on a co-oriented contact manifold preserving the contact structure then the contact quotient is topologically a stratified space (in the sense that a neighborhood of a point in the quotient is a product of a disk with a cone on a stratified space). As a corollary we obtain that symplectic quotients for proper Hamiltonian actions are topologically stratified spaces in this strong sense thereby extending and simplifying the results of [SL, BL].

1. INTRODUCTION

The notion of reduction of degrees of freedom in Hamiltonian mechanics has a long history. It was formalized in the early 1970s in the work of Marsden and Weinstein and, independently, Meyer. With the assumption of a freely acting symmetry group, they showed that the reduced system inherited a symplectic structure from the original system and that the reduced dynamics was intimately related to the original dynamics. A fair amount of effort by a number of people was spent during the 1980s on removing the assumption that the action is free. For example, it was proved by Arms, Marsden and Moncrief that the quotient of the zero level set of a moment map by the group action is a union of symplectic manifolds [AMM]. Arms, Cushman and Gotay showed that on a symplectic reduced space there is a natural Poisson algebra of functions [ACG]. In [SL, BL] it was proved that reduced spaces are stratified spaces in the sense that a neighborhood of a point is a product of a disk with a cone on a compact stratified space, that the strata are symplectic and that the symplectic structures on the strata are tightly related.

The earliest appearance of the notion of a contact quotient that we are aware of is in the work of Guillemin and Sternberg [GS] on homogeneous quantization, where it appeared in the guise of reduction of symplectic cones. Independently Albert [Al] and Geiges [Ge] each showed (under the assumption that the symmetry group acts freely on the zero level set of the contact moment map) that the quotient of the zero level set of the moment map by the group action, that is, the contact quotient, was naturally a contact manifold.

The first result of this paper removes the freeness assumption and shows that, in general, the contact quotients are naturally stratified spaces:

Theorem 1. *Let M be a manifold with a co-oriented contact structure ξ . Suppose a Lie group G acts properly on M preserving ξ . Choose a G -invariant contact form α with $\ker \alpha = \xi$ and let $\Phi : M \rightarrow \mathfrak{g}^*$ be the corresponding moment map.*

Then for every subgroup H of G , each connected component of the topological space

$$(M_{(H)} \cap \Phi^{-1}(0)) / G$$

is a manifold and the partition of the contact quotient

$$M_0 \equiv M // G := \Phi^{-1}(0) / G$$

into these manifolds is a stratification. The symbol $M_{(H)}$ stands for the set of points in M with the isotropy groups conjugate to H .

Partially supported by NSF grant DMS - 980305 and the American Institute of Mathematics. C. Willett was supported by a graduate VIGRE fellowship.

It can be shown that the strata of a contact quotient are contact manifolds. This will be discussed elsewhere.

The second main result of the paper is a natural short proof that symplectic quotients are stratified spaces. The proof uses the fact that neighborhoods of singularities of symplectic quotients are products of a disk with a cone on a contact reduced space. The result extends and simplifies the results of [SL, BL].

Theorem 2. *Let (M, ω) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and a corresponding equivariant moment map $\Phi : M \rightarrow \mathfrak{g}^*$. Fix a point $\beta \in \mathfrak{g}^*$; denote its isotropy group under the coadjoint action by G_β .*

Then for every subgroup H of G_β , each connected component of the topological space

$$(M_{(H)} \cap \Phi^{-1}(\beta)) / G_\beta$$

is a manifold, and the partition of the symplectic quotient at β

$$M // G(\beta) \equiv M_\beta := \Phi^{-1}(\beta) / G_\beta$$

into these manifolds is a stratification. The symbol $M_{(H)}$ stands for the set of points in M with the isotropy groups conjugate to H .

An outline of the contents of the paper is as follows:

1. Due to the multiple notions of stratification, we first make precise the meaning of the structure that we will place upon the contact and symplectic quotients.
2. A short review of the germane results of group actions on co-oriented contact manifolds is then pursued. We define moment maps, the contact quotient, and give a sketch of the result of [Ge].
3. We then prove the two main theorems simultaneously. The central idea is that both contact and symplectic quotients are locally modeled by reductions of vector spaces by compact groups.
4. We finish the paper with a proof of a local normal form theorem for group actions on contact manifolds which is modeled on the local normal form theorem of Marle and of Guillemin and Sternberg for symplectic manifolds.

Acknowledgments. The authors would like to thank Reyer Sjamaar for his helpful suggestion that an inductive argument on dimension of the links of singularities should quickly prove the main results of the paper.

A note on notation. Throughout the paper the Lie algebra of a Lie group denoted by a capital Roman letter will be denoted by the same small letter in the fraktur font: thus \mathfrak{g} denotes the Lie algebra of a Lie group G etc. The identity element of a Lie group is denoted by 1. The natural pairing between \mathfrak{g} and \mathfrak{g}^* will be denoted by $\langle \cdot, \cdot \rangle$.

When a Lie group G acts on a manifold M we denote the action by an element $g \in G$ on a point $x \in M$ by $g \cdot x$ or occasionally by $\tau_g(x)$; $G \cdot x$ denotes the G -orbit of x and so on. The vector field induced on M by an element X of the Lie algebra \mathfrak{g} of G is denoted by X_M . The isotropy group of a point $x \in M$ is denoted by G_x ; the Lie algebra of G_x is denoted by \mathfrak{g}_x and is referred to as the isotropy Lie algebra of x . We recall that $\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\}$.

If P is a principal G -bundle then $[p, m]$ denotes the point in the associated bundle $P \times_G M = (P \times M) / G$ which is the orbit of $(p, m) \in P \times M$.

If ω is a differential form on a manifold M and Y is a vector field on M , the contraction of ω by Y is denoted by $\iota(Y)\omega$.

2. PRELIMINARIES

2.1. Partitions and stratifications. Since the word “stratification” is used to describe many different types of partitions of topological spaces into manifolds, we start by describing precisely the meaning that we attach to the term. This is a quick review.

Definition 2.1. A **partition** of a Hausdorff topological space X is a collection of connected subsets $\{\mathcal{S}_i\}$ of X such that, set-theoretically, X is the disjoint union of \mathcal{S}_i 's. We refer to the pair $(X, \{\mathcal{S}_i\})$ as a **partitioned space** and to the subsets \mathcal{S}_i as **pieces**.

Remark 2.2. The connectedness of the pieces in the definition of a partitioned space is slightly superfluous as we could simply refine a partition by non-connected pieces into one by connected pieces.

Remark 2.3. The product of two partitioned spaces $(X, \{\mathcal{S}_i\})$ and $(Y, \{\mathcal{P}_j\})$ is the partitioned space $(X \times Y, \{\mathcal{S}_i \times \mathcal{P}_j\})$.

Remark 2.4. An open subset U of a partitioned space $(X, \{\mathcal{S}_i\})$ is naturally a partitioned space: the sets $U \cap \mathcal{S}_i$ form a partition of U .

Definition 2.5. Two partitioned spaces are **isomorphic** if they are homeomorphic and the homeomorphism takes pieces to pieces.

Recall that a cone on a topological space X , denoted by $\mathring{c}(X)$, is the quotient of the product $X \times [0, 1]$ by the relation $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. That is, $\mathring{c}(X)$ is $X \times [0, 1]$ with the “boundary” $X \times \{0\}$ collapsed to a point, the vertex $*$ of the cone.

Remark 2.6. The cone on a partitioned space $(X, \{\mathcal{S}_i\})$ is the partitioned space

$$(\mathring{c}(X), \{*\} \sqcup \{\mathcal{S}_i \times (0, 1)\})$$

Definition 2.7. If the pieces of a partitioned space $(X, \{\mathcal{S}_i\})$ are manifolds then we can define the dimension of X to be

$$\dim X = \sup_i \dim \mathcal{S}_i.$$

Remark 2.8. Note that the dimension of a partitioned space depends upon the choice of the partition. For example, every manifold admits a partition into points. For such a partition the dimension of the manifold as partitioned space is zero. However, the manifold dimension is always an upper bound for the partitioned space dimension.

We will only consider finite dimensional spaces. Unless otherwise mentioned we will partition connected manifolds into one piece only. Note that if a space X is partitioned into manifolds, then so is the cone on X and

$$\dim \mathring{c}(X) = \dim X + 1.$$

A stratification is a particularly nice type of a partition into manifolds. The definition is recursive on the dimension of partitioned space.

Definition 2.9. (cf. [GM]) A partitioned space $(X, \{\mathcal{S}_i\})$ is a **stratified space** if each piece \mathcal{S}_i is a manifold and if for every piece \mathcal{S} and for every point $x \in \mathcal{S}$ there exist

1. an open neighborhood U about x ,
2. an open ball B in \mathcal{S} about x ,
3. a compact stratified space L , called the **link** of the stratification of X at x (L may be empty) and
4. an isomorphism $\varphi : B \times \mathring{c}(L) \rightarrow U$ of partitioned spaces (If $L = \emptyset$ we require that U is homeomorphic to the ball B .)

The pieces of a stratified space are called **strata**; the collection of strata is called a **stratification**.

Remark 2.10. 1. A zero dimensional stratified space is a discrete set of points.

2. A connected manifold is a stratified space with exactly one stratum.

3. A cone on a compact manifold is a stratified space.

An example of a stratification is furnished by the orbit type partition of an orbit space for a proper action of a Lie group on a manifold. Let us briefly review the relevant ideas (see the introduction for the notational conventions).

An action of a Lie group G on a manifold M is **proper** if the map $G \times M \rightarrow M \times M$ given by $(g, m) \mapsto (g \cdot m, m)$ is proper, i.e., the preimages of compact sets under this map are compact. Note that for proper actions all isotropy groups are compact.

A **slice** for an action of a Lie group G on a manifold M at a point x is a G_x invariant submanifold Σ such that $G \cdot \Sigma$ is an open subset of M and such that the map $G \times \Sigma \rightarrow G \cdot \Sigma$, $(g, s) \mapsto g \cdot s$ descends to a diffeomorphism $G \times_{G_x} \Sigma \rightarrow G \cdot \Sigma$, $[g, s] \mapsto g \cdot s$. Thus for any point $y \in G \cdot \Sigma$, the orbit $G \cdot y$ intersects the slice Σ in a single G_x -orbit. Also, for any $y \in \Sigma$, $G_y \subset G_x$. A theorem of Palais [P] asserts that for smooth proper actions slices exist at every point.

Let M be a manifold with a proper action of a Lie group G . For a subgroup H of G denote by $M_{(H)}$ the set of points in M whose stabilizer is conjugate to H ; i.e.,

$$M_{(H)} = \{x \in M \mid \text{there exists } g \in G \text{ such that } gG_xg^{-1} = H\},$$

where G_x denotes the stabilizer of x . The set $M_{(H)}$ is often referred to as the set of points of **orbit type** H .

It is an easy consequence of the existence of slices that the connected components of the sets $M_{(H)}$ are manifolds and that the components of the quotient $M_{(H)}/G$ are manifolds as well. Moreover, using induction on the dimension of M/G one can show that the partition of M/G into the connected components of the sets of the form $M_{(H)}/G$ is a stratification. The point of this paper is to adapt the argument for contact and symplectic quotients.

Note that in the paper we refer to both partitions

$$M = \bigsqcup_{(H)} M_{(H)}$$

and

$$M/G = \bigsqcup_{(H)} M_{(H)}/G$$

as stratifications by orbit type.

Remark 2.11. If H and K are conjugate subgroups of G , then $M_{(H)} = M_{(K)}$. Thus, the indexing set for the stratification by orbit type is the set of conjugacy class of G .

We will extensively use the following easy fact.

Lemma 2.12. *Suppose a Lie group G acts properly on a manifold M . Let $x \in M$ be a point and let $\Sigma \ni x$ be a slice through x for the action of G . Denote the isotropy group of x by H .*

Then for any H -invariant subset Z of Σ

$$(G \cdot Z)/G = Z/H \quad (\text{as partitioned spaces})$$

where the left hand side is partitioned by G -orbit types and the right hand by H -orbit types.

Proof. Since Σ is a slice, for any $m \in Z \subset \Sigma$, $G \cdot m \cap Z$ is a single H -orbit. □

2.2. Group actions on contact manifolds. Recall that a (co-oriented) **contact structure** on a manifold M of dimension $2n + 1$ is a distribution ξ on M given globally as the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$. The form α is called a **contact form**. Whenever convenient we will refer to a pair (M, α) or to a pair (M, ξ) as a contact manifold.

An action of a Lie group G on a manifold preserving a contact structure ξ is called a **contact action**.

Remark 2.13. If a Lie group G acts properly on a manifold and preserves a co-oriented contact structure ξ , then there exists a G -invariant 1-form α with $\ker \alpha = \xi$. If G is compact, the proof of this assertion is easy: since ξ is co-oriented, there exists by definition a 1-form α_0 with $\ker \alpha_0 = \xi$. If α_0 is not G -invariant, average it over G . If G is not compact, the argument is only slightly more complicated. One adopts Palais's proof of the existence of invariant Riemannian metrics on manifolds with proper group actions [P] to "average" α_0 . See, for example, [L] for details.

Definition 2.14. Suppose a Lie group G acts on a manifold M preserving a contact form α . We define the **contact moment map** $\Phi : M \rightarrow \mathfrak{g}^*$ by

$$\langle \Phi(x), X \rangle = \alpha_x(X_M(x))$$

for all X in the Lie algebra \mathfrak{g} of G and all $x \in M$. The contact moment map is G -equivariant, where as usual G acts on \mathfrak{g}^* by the coadjoint action (see [Ge]). Hence, the G action descends to an action on the zero level set and we define the **contact quotient** (or **contact reduction**) of M by G to be the topological space

$$M_0 \equiv M//G := \Phi^{-1}(0)/G.$$

We will use the symbols $M//G$ and M_0 interchangeably.

We define the **canonical partition** of the quotient $M//G$ to be the the connected components of the sets of the form

$$(M_{(H)} \cap \Phi^{-1}(0))/G$$

for all conjugacy classes (H) of G .

Similarly symplectic quotients also have a canonical partition.

Definition 2.15. Let (M, ω) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and a corresponding equivariant moment map $\Phi : M \rightarrow \mathfrak{g}^*$. We define the symplectic quotient of M at a point $\beta \in \mathfrak{g}^*$ to be the topological space

$$M//G(\beta) \equiv M_\beta := \Phi^{-1}(\beta)/G_\beta,$$

where G_β denotes the isotropy group of β under the coadjoint action.

Next we define the **canonical partition** of the quotient M_β to be the connected components of the sets of the form

$$(M_{(H)} \cap \Phi^{-1}(\beta))/G_\beta$$

for all conjugacy classes (H) of G .

Remark 2.16. Suppose a Lie group G acts properly on a manifold M preserving a contact form α . Albert and, independently, Geiges showed that if G acts freely on $\Phi^{-1}(0)$, then 0 is a regular value of the corresponding moment map Φ , the contact quotient M_0 is a manifold and $\alpha|_{\Phi^{-1}(0)}$ descends to a contact form α_0 on M [Al, Ge]. See also [GS] where the result was obtained earlier in a different form (cf. the introduction).

Remark 2.17. Recall that the **symplectization** of a contact manifold (M, α) is the symplectic manifold $(M \times \mathbb{R}, d(e^t \alpha))$. If a Lie group G acts on M and preserves α then the trivial extension of the action of G to $M \times \mathbb{R}$ is Hamiltonian and a corresponding moment map $\Psi : M \times \mathbb{R} \rightarrow \mathfrak{g}^*$ is related to the contact moment map $\Phi : M \rightarrow \mathfrak{g}^*$ by the formula

$$\Psi(m, t) = -e^t \Phi(m) \quad \text{for all } (m, t) \in M \times \mathbb{R}.$$

In particular, $\Psi^{-1}(0) = \Phi^{-1}(0) \times \mathbb{R}$. One can use this to give an alternative proof that, in the case of free proper actions, contact quotients are contact manifolds.

Remark 2.18. Recall that the **contactization** of an exact symplectic manifold $(M, d\lambda)$ is the contact manifold $(M \times \mathbb{R}, \lambda + dt)$. If a Lie group G acts on M preserving the 1-form λ then the action is Hamiltonian with moment map $\Phi : M \rightarrow \mathfrak{g}^*$ given by $\langle \Phi(m), X \rangle = -\lambda_m(X_M(m))$ for all $X \in \mathfrak{g}$, all $m \in M$. The trivial extension of the action of G to $M \times \mathbb{R}$ preserves the contact form. By definition the contact moment map $\Psi : M \times \mathbb{R} \rightarrow \mathfrak{g}^*$ is given by $\langle \Psi(m, t), X \rangle = -\lambda_m(X_M(m))$. Thus Ψ factors as $\Psi = \Phi \circ \text{pr}$, where $\text{pr} : M \times \mathbb{R} \rightarrow M$ is the projection. It follows that, as spaces,

$$(M \times \mathbb{R})//G = M//G(0) \times \mathbb{R}.$$

If G acts freely on $\Phi^{-1}(0)$, it is not hard to show that the symplectic quotient at zero $M//G(0)$ is an exact symplectic manifold and that the two sides are the same as contact manifolds.

It will be particularly useful for us to consider the contactization of symplectic vector spaces. Suppose (V, ω_V) is a symplectic vector space. The Lie derivative of ω_V with respect to the radial vector field $R(v) := v$ is $2\omega_V$, hence

$$d\left(\frac{1}{2}\iota(R)\omega_V\right) = \omega_V.$$

Therefore $\frac{1}{2}\iota(R)\omega_V + dt$ is a contact form on $V \times \mathbb{R}$. Note that if $\rho : H \rightarrow \text{Sp}(V, \omega_V)$ is a symplectic representation of a Lie group H then, since the action of H preserves the radial vector field R , the 1-form $\frac{1}{2}\iota(R)\omega_V$ is H -invariant. Hence if we trivially extended the action of H to the contactization $V \times \mathbb{R}$, the contact form $\frac{1}{2}\iota(R)\omega_V + dt$ is H invariant as well. We will refer to the contact manifold $(V \times \mathbb{R}, \frac{1}{2}\iota(R)\omega_V + dt)$ as a **contact vector space** and as **the** contactization of (V, ω_V) .

3. PROOF OF THE MAIN RESULT

Our proof of the main result — Theorems 1 and 2 — has two ingredients. The first one is an observation that arbitrary contact quotients are modeled on the contact quotients of contact vector spaces and that arbitrary symplectic quotients are modeled on the symplectic quotients of symplectic vector spaces. The second one is Lemma 3.3, below, which describes the structure of contact and symplectic quotients of vector spaces.

The idea that a symplectic quotient for a reasonable group action can be modeled on a symplectic quotient of a vector space is due to R. Sjamaar. The idea was implemented for reduction at zero by an action of a compact Lie group in [SL] and for reduction at a locally closed coadjoint orbit by a proper action of an arbitrary Lie group in [BL].

It was pointed out by Ortega and Ratiu that the technical assumption on the coadjoint orbit in [BL] is unnecessary if one defines symplectic quotients as in Definition 2.15. It was shown in [BL] that if the coadjoint orbit $G \cdot \beta$ is locally closed then the sets $(M_{(H)} \cap \Phi^{-1}(G \cdot \beta))/G \simeq (M_{(H)} \cap \Phi^{-1}(\beta))/G_\beta$ are symplectic manifolds and that the partition of $M_{G \cdot \beta} := \Phi^{-1}(G \cdot \beta)/G \simeq M_\beta$ by these manifolds is a stratification. Ortega proved that one can drop the assumption on the coadjoint orbit and still show that these sets are symplectic manifolds, that they are locally closed in the symplectic quotient and that the canonical partition is locally finite [O] (see also [OR]). Theorem 1 asserts that the canonical partition is in fact a stratification. Our proof of it does not directly use Ortega's result. Rather, it uses the following two propositions, which are variations on an argument in [SL].

Proposition 3.1. *Let (M, ω) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and a corresponding equivariant moment map $\Phi : M \rightarrow \mathfrak{g}^*$ and let $M_\beta := \Phi^{-1}(\beta)/G$ be the symplectic quotient at $\beta \in \mathfrak{g}^*$. For any point $x \in M_\beta$ there exist a neighborhood \mathcal{U} of x in M_β , a symplectic vector space V with a symplectic action of a compact Lie group H , a neighborhood \mathcal{U}_0 of the image of 0 in the symplectic quotient V_0 and a homeomorphism $\varphi : \mathcal{U} \rightarrow \mathcal{U}_0$ which maps the pieces of the canonical partition of $M_\beta \cap \mathcal{U}$ into pieces of the canonical partition of $V_0 \cap \mathcal{U}_0$.*

Proposition 3.2. *Suppose a Lie group G acts properly on a manifold M preserving a contact form α . Let $\Phi : M \rightarrow \mathfrak{g}^*$ be the corresponding moment map. For any point x in the contact quotient $M//G := \Phi^{-1}(0)/G$ there exist a neighborhood \mathcal{U} of x in $M//G$, a contact vector space V with a contact action of a compact Lie group H , a neighborhood \mathcal{U}_0 of the image of 0 in a contact quotient $V//H$ and a homeomorphism $\varphi : \mathcal{U} \rightarrow \mathcal{U}_0$ which maps the pieces of the canonical partition of $M_\beta \cap \mathcal{U}$ into pieces of the canonical partition of $(V//H) \cap \mathcal{U}_0$.*

We postpone the proof of these two results for a later section and proceed with the study of the quotients of vector spaces:

Lemma 3.3. *Let $\rho : H \rightarrow \mathrm{Sp}(V, \omega)$ be a symplectic representation of a compact Lie group H .*

1. *The symplectic quotient at zero $V//H(0)$ is isomorphic, as a partitioned space, to the product of a symplectic vector space U and a cone on the quotient of a standard contact sphere S :*

$$V//H(0) = U \times \mathring{c}(S//H).$$

2. *The contact quotient of the contactization of V is isomorphic, as a partitioned space, to the product of the contact vector space $U \times \mathbb{R}$ and a cone on the quotient of the standard contact sphere S :*

$$(V \times \mathbb{R})//H = (U \times \mathbb{R}) \times \mathring{c}(S//H).$$

Proof. Let U be the vector subspace of H -fixed vectors in V ; it is a symplectic subspace of V . Let W denote the symplectic perpendicular to U . It is a symplectic H -invariant subspace as well. Let R as before denote the radial vector field on V . Since $d(\frac{1}{2}\iota(R)\omega) = \omega$, the moment map Φ_V for the action of H on V is given by the formula

$$\langle \Phi(v), X \rangle = \left(\frac{1}{2}\iota(R)\omega\right)(X_V(v))$$

for all $v \in V$ and all X in the Lie algebra \mathfrak{h} of H , where X_V denotes the linear vector field induced by X on V . Hence $\Phi^{-1}(0) = U \times \Phi_W^{-1}(0)$ where Φ_W is the restriction of Φ to W . Note that Φ_W is also a moment map for the action of H on W .

Since H is compact there exists an H invariant complex structure J on W compatible with $\omega_W := \omega|_W$. Let S denote the unit sphere with respect to the inner product $\omega_W(\cdot, J\cdot)$. Since Φ_W is homogeneous, $\Phi_W^{-1}(0) = \mathring{c}(S \cap \Phi_W^{-1}(0))$. Since the action of H is linear, $\Phi_W^{-1}(0)/H = U \times \mathring{c}((S \cap \Phi_W^{-1}(0))/H)$. Note that the restriction of $\frac{1}{2}\iota(R)\omega_W$ to S is a contact form and that the restriction of Φ_W to S is the corresponding contact moment map. Therefore the space $(S \cap \Phi_W^{-1}(0))/H$ is the contact quotient $S//H$ and the first part of the lemma follows.

The second part of the lemma follows as well since the contact moment map Ψ on the contactization $V \times \mathbb{R}$ is related to the symplectic moment map Φ on V by $\Psi(v, t) = \Phi(v)$ for all $(v, t) \in V \times \mathbb{R}$ (see Remark 2.17). \square

Remark 3.4. If in the Lemma above the representation ρ is trivial, then $V//H(0) = V$ and $(V \times \mathbb{R})//H = V \times \mathbb{R}$. In particular, if $V = \{0\}$, then $(V \times \mathbb{R})//H = \mathbb{R}$.

Proof of Theorems 1 and 2. Lemma 3.3 together with Proposition 3.2 imply that a contact quotient is locally isomorphic as a partitioned space to a product of an odd-dimensional disk and a cone on the contact quotient of a standard contact sphere. It follows that the pieces of the canonical decomposition of a contact quotient are odd dimensional manifolds and that they are locally closed.

Similarly Lemma 3.3 together with Proposition 3.1 imply that a symplectic quotient is locally isomorphic as a partitioned space to a product of an even dimensional disk and a cone on the contact quotient of a standard contact sphere. It follows that the pieces of the canonical decomposition of a symplectic quotient are even dimensional manifolds.

We next argue by induction on the dimension of contact quotients that contact quotients are stratified spaces. The smallest dimension that a contact quotient can have is one. If it is one dimensional then it is a one dimensional manifold (which need not be connected).

Assume now that the dimension of our contact quotient X is bigger than one and that any contact quotient Z with $\dim Z < \dim X$ is a stratified space. By Proposition 3.2 a neighborhood of a point in X is isomorphic, as a partitioned space, to a neighborhood of a point in the contact quotient of a contact vector space. By Lemma 3.3, the contact quotient of a contact vector space is isomorphic to the product of a contact vector space and a cone on the contact quotient of a sphere. By induction, the contact quotient of the sphere is a stratified space. Therefore X is a stratified space by definition.

Finally, by Proposition 3.1 and Lemma 3.3 a symplectic quotient is isomorphic to a product of a disk and a cone on the contact quotient of a contact sphere. By the previous paragraph, the contact quotient of the sphere is a stratified space. Therefore a symplectic quotient is also a stratified space. \square

Observe that we have, in fact, proved the following.

Theorem 3.5. *Let $(X, \{\mathcal{S}_\alpha\})$ be a partitioned space with the property that a neighborhood of every point is isomorphic (as partitioned spaces) to a product of a disk with the cone on the contact quotient of a standard contact sphere. Then the partition $\{\mathcal{S}_\alpha\}$ is a stratification.*

We finish the section with a short proof that connected contact and symplectic quotients have unique connected open dense strata. In particular we recover Theorem 5.9 of [SL]. The proof has two parts. We first remark that the contact quotient of a standard contact sphere by a linear action of a compact Lie group is connected.

Lemma 3.6. *Let $S^{2n-1} \subset \mathbb{C}^n$ be the standard contact sphere and let K be a closed subgroup of the unitary group $U(n)$. Then the contact quotient S^{2n-1}/K is connected.*

Proof. The map $f : S^{2n-1} \rightarrow \mathbb{R}$, $f(z) = \frac{1}{2}||z||^2$ is a moment map for the action $(\lambda, z) \mapsto \lambda z$ of the circle $S^1 = U(1)$ on \mathbb{C}^n , and the sphere S^{2n-1} is a level set of f . Since the action of S^1 commutes with the action of K , the restriction of the K -moment map $\Phi : \mathbb{C}^n \rightarrow \mathfrak{k}^*$ to S^{2n-1} descends to a moment map $\bar{\Phi}$ for the action of K on the symplectic quotient $f^{-1}(2)/S^1 = S^{2n-1}/S^1 = \mathbb{C}P^{n-1}$. Since the projective space $\mathbb{C}P^{n-1}$ is compact and connected, the fibers of the moment map $\bar{\Phi} : \mathbb{C}P^{n-1} \rightarrow \mathfrak{k}^*$ are connected by a theorem of Kirwan (see [K2, Remark 2.1] and [K1, Remark 9.1]).

On the other hand, the restriction $\Phi|_{S^{2n-1}}$ is the contact moment map for the action of K on the sphere S^{2n-1} . Since the fibers of $\bar{\Phi}$ are S^1 quotients of the fibers of $\Phi|_{S^{2n-1}}$, the connectedness of the fibers of $\bar{\Phi}$ implies that the fibers of $\Phi|_{S^{2n-1}}$ are connected as well. Therefore $S^{2n-1}/K = (\Phi|_{S^{2n-1}})^{-1}(0)/K$ is connected. \square

The second part of the proof is:

Proposition 3.7. *Let $(X, \{\mathcal{S}_i\})$ be a connected stratified space. Assume, recursively, that all links are connected. That is, the link of every point is connected, the link of every point in the link is connected and so on. Then there is a unique open dense stratum X^r in X .*

Proof. The proof is an induction on the dimension of X . Let X^r be the union of all the open strata in X . We show first that X^r is dense. Using density, we then show that X^r is connected and hence consists of a single stratum. Note that a point in a stratified space has an empty link if and only if it lies in an open stratum. This implies that if a stratum contains a set which is open in X then the whole stratum is open as well.

Let x be a point in X and \mathcal{S} be the stratum containing x . By definition there is an open neighborhood U of x in X , an open ball B in \mathcal{S} , and an isomorphism $\rho : \mathring{c}(L) \times B \rightarrow U$ of partitioned spaces, where L is the link of x . If the link is empty, then $x \in X^r$. Otherwise, by induction L contains a unique open dense stratum L^r . Then $L^r \times (0, 1) \times B$ is open in $\mathring{c}(L) \times B$. Let \mathcal{S}' denote the stratum in X with $\rho(L^r \times (0, 1) \times B) = \mathcal{S}' \cap U$. Then $\mathcal{S}' \cap U$ is open in U . Therefore \mathcal{S}' is open in X , hence $\mathcal{S}' \subset X^r$. Clearly x lies in the closure of \mathcal{S}' . Therefore x is in the closure of X^r . This proves that X^r is dense.

We now prove that X^r is connected. Suppose not. Choose disjoint open sets U_1 and U_2 in X with $U_1 \cup U_2 = X^r$. By density, $X = \bar{U}_1 \cup \bar{U}_2$. Because X is connected, however, $\bar{U}_1 \cap \bar{U}_2 \neq \emptyset$. Choose $y \in \bar{U}_1 \cap \bar{U}_2$. Then $y \notin X^r$ and so the link L_y of y is non-empty. Hence, by induction, there is a unique open stratum L_y^r in L_y . Choose a neighborhood, U of y of the form $U = \dot{c}(L_y) \times B$, where B is a ball. On the one hand, $X^r \cap U = L_y^r \times (0, \epsilon) \times B$ is connected. On the other, $X^r \cap U = (U_1 \cap U) \cap (U_2 \cap U)$ is not. \square

Remark 3.8. We can remove the hypothesis that the space X is connected. In this case, we work component by component to conclude that each connected component of X has a unique connected open dense stratum.

Putting Lemma 3.6 and Proposition 3.7 together and using the proof of Theorems 1 and 2, we obtain:

Theorem 3.9. *Suppose a Lie group G acts properly on a manifold M preserving a contact form α . Then each connected component of the contact quotient $M//G$ has a unique connected open dense stratum.*

Let (M, ω) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and a corresponding equivariant moment map $\Phi : M \rightarrow \mathfrak{g}^$. Fix a point $\beta \in \mathfrak{g}^*$; denote its isotropy group under the coadjoint action by G_β . Then for every point $\beta \in \mathfrak{g}^*$ a connected component of the symplectic quotient $M//G(\beta)$ has a unique connected open dense stratum.*

4. LOCAL MODELS OF SYMPLECTIC AND CONTACT QUOTIENTS

The main tool for proving Proposition 3.2 is

Theorem 4.1. *Suppose a Lie group G acts properly on a manifold M preserving a contact form α and let $\Phi : M \rightarrow \mathfrak{g}^*$ be the corresponding moment map. Let $x \in \Phi^{-1}(0)$ be a point, let H be its isotropy group. Denote the Lie algebra of H by \mathfrak{h} and the annihilator of \mathfrak{h} in \mathfrak{g}^* by \mathfrak{h}° . Choose an H -equivariant splitting $\mathfrak{g}^* = \mathfrak{h}^\circ \oplus \mathfrak{h}^*$; let $i : \mathfrak{h}^* \rightarrow \mathfrak{g}^*$ be the corresponding injection.*

There exists a G -invariant neighborhood U of x in M , a G -invariant neighborhood U_0 of the zero section on the vector bundle $\mathcal{Y} = (G \times_H (\mathfrak{h}^\circ \times W)) \rightarrow G/H$ and a G -equivariant diffeomorphism $\phi : U_0 \rightarrow U$ such that $\phi([1, 0, 0]) = x$ and

$$(\Phi \circ \phi)([g, \eta, w]) = f([g, \eta, v]) \text{Ad}^\dagger(g) (\eta + i(\Phi_W(w))),$$

where

1. $\text{Ad}^\dagger : G \rightarrow GL(\mathfrak{g}^*)$ is the coadjoint representation,
2. W is the contactization of the maximal symplectic subspace V of the symplectic vector space $(\ker \alpha_x, d\alpha_x)$ complementary to the tangent space $T_x(G \cdot x)$; V can be and is chosen to be H -invariant;
3. $\Phi_W : W \rightarrow \mathfrak{h}^*$ is the moment map for the linear action of H on the contact vector space W ;
4. f is a nowhere vanishing function;
5. G acts on \mathcal{Y} by $g \cdot [a, \eta, v] = [ga, \eta, v]$.

We postpone the proof of this Theorem till the last section of the paper and proceed with the proof of Proposition 3.2.

Proof of Proposition 3.2. Define $F : \mathcal{Y} \rightarrow \mathfrak{g}^*$ by $F([g, \eta, w]) = \text{Ad}^\dagger(g) (\eta + i(\Phi_W(w)))$. Since the function f is nonvanishing the map ϕ^{-1} sends $\Phi^{-1}(0) \cap U$ equivariantly and homeomorphically onto the set $F^{-1}(0) \cap U_0$. Hence $(\Phi^{-1}(0) \cap U)/G = (F^{-1}(0) \cap U_0)/G$ as partitioned spaces (the partitioning is by G -orbit type). Therefore it is enough to show that

$$(4.1) \quad F^{-1}(0)/G = \Phi_W^{-1}(0)/H \equiv W//H \quad \text{as partitioned spaces.}$$

Next note that the vector space $\mathfrak{h}^\circ \times W$ embeds canonically into \mathcal{Y} by $(\eta, w) \mapsto [1, \eta, w]$ and that $\mathfrak{h}^\circ \times W$ is a slice through $[1, 0, 0]$ for the action of G on \mathcal{Y} .

Because $\mathfrak{h}^\circ \oplus i(\mathfrak{h}^*) = \mathfrak{g}^*$ and $i : \mathfrak{h}^* \rightarrow \mathfrak{g}^*$ is injective, $\eta + i(\Phi_W(w)) = 0$ if and only if $\eta = 0$ and $\Phi_W(w) = 0$. Therefore

$$F^{-1}(0) = \{[g, \eta, w] \mid \eta = 0 \text{ and } \Phi_W(w) = 0\} = G \times_H (\{0\} \times \Phi_W^{-1}(0)).$$

Hence $F^{-1}(0) = G \cdot \Phi_W^{-1}(0)$. Therefore by Lemma 2.12 equation (4.1) holds. \square

Proof of Proposition 3.1. The proof is essentially the same as that of Proposition 3.2 above. Since we are not reducing as zero, it is a little more delicate. We use a version of the local normal form theorem due to Marle and to Guillemin and Sternberg which is proved on pp. 212 – 215 of [BL]:

Theorem 4.2. *Let (M, ω) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and a corresponding equivariant moment map $\Phi : M \rightarrow \mathfrak{g}^*$. Let $x \in M$ be a point. Let H denote its isotropy group. Let $\beta = \Phi(x)$ and let G_β denote the isotropy group of β . Define*

$$V := T_x(G \cdot x)^\omega / (T_x(G \cdot x)^\omega \cap T_x(G \cdot x)),$$

where $T_x(G \cdot x)^\omega$ denotes the symplectic perpendicular to the tangent space to the orbit $T_x(G \cdot x)$ in $T_x M$; it is a symplectic vector space with a linear symplectic action of H .

Let \mathfrak{g} , \mathfrak{g}_β and \mathfrak{h} denote the Lie algebras of G , G_β and H , respectively. Let \mathfrak{h}° denote the annihilator of \mathfrak{h} in \mathfrak{g}_β^* , and \mathfrak{g}_β° the annihilator of \mathfrak{g}_β in \mathfrak{g}^* . Choose an H -equivariant splitting $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{h}^\circ \oplus \mathfrak{g}_\beta^\circ$; let $i : \mathfrak{h}^* \rightarrow \mathfrak{g}_\beta^*$ and $j : \mathfrak{g}_\beta^* \rightarrow \mathfrak{g}^*$ denote the corresponding injections.

There exists a G -invariant neighborhood U of x in M , a G -invariant neighborhood U_0 of the zero section on the vector bundle $\mathcal{Y} = (G \times_H (\mathfrak{h}^\circ \times V)) \rightarrow G/H$ and a G -equivariant diffeomorphism $\phi : U \rightarrow U_0$ such that

$$(\Phi \circ \phi)([g, \eta, v]) = \text{Ad}^\dagger(g) (\beta + j(\eta + i(\Phi_V(v)))) ,$$

where

1. $\text{Ad}^\dagger : G \rightarrow GL(\mathfrak{g}^*)$ is the coadjoint representation and
2. $\Phi_V : V \rightarrow \mathfrak{h}^*$ is the moment map for the action of H on V .

Define $F : \mathcal{Y} \rightarrow \mathfrak{g}^*$ by $F([g, \eta, v]) = \text{Ad}^\dagger(g) (\beta + j(\eta + i(\Phi_V(v))))$. The map ϕ sends $\Phi^{-1}(\beta) \cap U$ homeomorphically and G -equivariantly onto $F^{-1}(\beta) \cap U_0$. Hence $(\Phi^{-1}(0) \cap U)/G_\beta = (F^{-1}(0) \cap U_0)/G_\beta$ as spaces partitioned by G -orbit types. Therefore it is enough to prove that for some sufficiently small G_β -invariant neighborhood \mathcal{O} of $[1, 0, 0]$ in \mathcal{Y}

$$(4.2) \quad (F^{-1}(0) \cap \mathcal{O})/G_\beta = (\Phi_V^{-1}(0) \cap \tilde{\mathcal{O}})/H$$

as partitioned spaces for some H -invariant neighborhood $\tilde{\mathcal{O}}$ of 0 in V , where the left hand side is partitioned by G -orbit types and the right hand side is partitioned by H -orbit types.

Note that the canonical embedding of the vector space $\mathfrak{h}^\circ \times V$ into \mathcal{Y} , $(\eta, v) \mapsto [1, \eta, v]$ makes $\mathfrak{h}^\circ \times V$ into a slice at $[1, 0, 0]$ for the action of G on \mathcal{Y} . The vector space $\mathfrak{h}^\circ \times V$ is also a slice at $[1, 0, 0]$ for the action of G_β on $G_\beta \times_H (\mathfrak{h}^\circ \times V)$. Therefore $\mathcal{Y}/G = (\mathfrak{h}^\circ \times V)/H = (G_\beta \times_H (\mathfrak{h}^\circ \times V))/G_\beta$. It follows from Lemma 2.12 that for any subgroup K of G_β

$$(4.3) \quad (G_\beta \times_H (\mathfrak{h}^\circ \times V))_{(K)} = Y_{(K)} \cap (G_\beta \times_H (\mathfrak{h}^\circ \times V)).$$

Conversely if $Y_{(K)} \cap (G_\beta \times_H (\mathfrak{h}^\circ \times V)) \neq \emptyset$, we may assume that $K \subset G_\beta$. Then equation (4.3) holds. Therefore the partition of the left hand side of (4.2) by G -orbit types is the same as its partition by G_β -orbit types.

Recall that the tangent space to the coadjoint orbit through β is canonically isomorphic to the annihilator of \mathfrak{g}_β in \mathfrak{g}^* : $T_\beta(\text{Ad}^\dagger(G)\beta) \simeq \mathfrak{g}_\beta^\circ$. Since $\mathfrak{g}^* = j(\mathfrak{g}_\beta^*) \oplus \mathfrak{g}_\beta^\circ$, the vector bundle $G \times_{G_\beta} \mathfrak{g}_\beta^*$ is the normal bundle of the orbit $\text{Ad}^\dagger(G)\beta$ in \mathfrak{g}^* . Consider the map $\mathcal{E} : G \times_{G_\beta} \mathfrak{g}_\beta^* \rightarrow \mathfrak{g}^*$ given by $\mathcal{E}([g, \eta]) = \text{Ad}^\dagger(g)(\beta + j(\eta))$. It is the exponential map for a flat H -invariant metric on \mathfrak{g}^* . The differential $d\mathcal{E}$ is an isomorphism at the point $[1, 0]$. Therefore \mathcal{E} is an open embedding

on a sufficiently small neighborhood \mathcal{O}' of $[1, 0]$. In particular $\mathcal{O}' \cap \mathcal{E}^{-1}(\beta) = \{[1, 0]\}$. We now factor F as follows: let

$$F_2 : G \times_H (\mathfrak{h}^\circ \times V) \rightarrow G \times_H \mathfrak{g}_\beta^*, \quad [g, \eta, v] \mapsto [g, \eta + i(\Phi_W(v))],$$

$$F_1 : G \times_H \mathfrak{g}_\beta^* \rightarrow G \times_{G_\beta} \mathfrak{g}_\beta^*, \quad [g, \eta] \mapsto [g, \eta].$$

We have $F = \mathcal{E} \circ F_1 \circ F_2$. Let $\mathcal{O} = (F_1 \circ F_2)^{-1}(\mathcal{O}')$. Then $F^{-1}(\beta) \cap \mathcal{O} = (F_1 \circ F_2)^{-1}(\mathcal{O}' \cap \mathcal{E}^{-1}(\beta)) = (G_\beta \times_H (\{0\} \times \Phi_V^{-1}(0))) \cap \mathcal{O}$. Consequently by Lemma 2.12 $(F^{-1}(\beta) \cap \mathcal{O})/G_\beta = (\Phi_V^{-1}(0) \cap \tilde{\mathcal{O}})/H$ as partitioned spaces, where the left hand side is partitioned by G_β -orbit types, the right hand side is partitioned by H -orbit types and where $\tilde{\mathcal{O}} = V \cap \mathcal{O}$. But we have seen that the partition of $G_\beta \times_H (\mathfrak{h}^\circ \times V)$ by G -orbit types and by G_β orbit types is the same partition. The Proposition now follows. \square

5. CONTACT LOCAL NORMAL FORM

In this section we provide a proof of the local normal form theorem (Theorem 4.1) that allowed us to establish our main results (Theorems 1 and 2). Not surprising, this requires an equivariant Darboux theorem for contact manifolds, the proof of which is the standard deformation argument of Moser. We then review the contact analogues of isotropic submanifolds and symplectic normal bundles, introduced by Weinstein in [We]. We finish with the proof of the local normal form theorem.

Recall that the Reeb vector field Y on a contact manifold (M, α) is the unique vector field satisfying $\iota(Y)\alpha = 1$ and $\iota(Y)d\alpha = 0$. If a Lie group acts on M preserving α then by uniqueness the Reeb vector field is invariant.

Observe that an equivariant version of Gray's stability theorem holds.

Theorem 5.1. (*Equivariant Gray's Stability Theorem*) *Suppose a Lie group G acts on a contact manifold M and $\{\alpha_t\}, t \in [0, 1]$ is a smoothly varying family of invariant contact forms. Suppose that $N \subseteq M$ is a invariant submanifold and that*

$$\frac{d}{dt}\alpha_t(x) = \frac{d}{dt}d\alpha_t(x) = 0$$

for all $x \in N$. Then there is a family of equivariant diffeomorphisms $\{\psi_t\}$ of M with $\psi_t|_N = \text{Id}$ and $\psi_t^(\alpha_t) = f_t\alpha_0$ for all t and for some family of positive functions $\{f_t\}$.*

As a consequence of Gray's stability theorem we get:

Theorem 5.2. (*Equivariant Relative Contact Darboux*) *Let a Lie group G act properly on a manifold M preserving an embedded submanifold N . Suppose α and β are two invariant contact forms with*

$$\begin{aligned} \alpha(x) &= \beta(x) \\ d\alpha(x) &= d\beta(x) \end{aligned}$$

for all $x \in N$. Then there are invariant open neighborhoods U_0, U_1 of N , an equivariant diffeomorphism $\psi : U_0 \rightarrow U_1$ fixing N , and a positive function f such that $\psi^(\beta) = f\alpha$.*

Proof. For $t \in [0, 1]$, define $\gamma_t = (1-t)\alpha + t\beta$. For all $x \in N$ and all t , we have $\gamma_t(x) = \alpha(x) = \beta(x)$ and $d\gamma_t(x) = d\alpha(x) = d\beta(x)$. Hence, γ_t is contact in an open neighborhood of N . Since γ_t is invariant, Gray's theorem applies. The time 1 map of the isotopy ψ_t exists on some open set O of N since it exists on a neighborhood of each point of N in M . Set $U_0 = G \cdot O$ and $U_1 = \psi_1(U_0)$. \square

Definition 5.3. Let (M, α) be a contact manifold. A submanifold L of M is *isotropic* if it is tangent to the contact distribution $\ker \alpha$, that is, $T_x L \subset \ker \alpha_x$ for every $x \in L$.

Example 5.4. Suppose a Lie group G acts properly on a manifold M preserving a contact form α . Then orbits $G \cdot x$ which lie in the zero level set of the contact moment Φ are isotropic submanifolds by definition of the moment map.

Remark 5.5. If L is an isotropic submanifold of a contact manifold (M, α) then for every point $x \in L$ the tangent space $T_x L$ is an isotropic subspace of the symplectic vector space $(\ker \alpha_x, d\alpha_x)$, hence the terminology.

Definition 5.6. Let L be an isotropic submanifold of a contact manifold (M, α) . Then for every point $x \in L$ the quotient $(T_x L)^{d\alpha}/T_x L$ is a symplectic vector space, where $(T_x L)^{d\alpha}$ denotes the symplectic perpendicular to $T_x L$ in $\ker \alpha_x$ with respect to $d\alpha$ ($T_x L \subset (T_x L)^{d\alpha}$ by Remark 5.5). The *symplectic normal bundle* of L in (M, α) is the vector bundle

$$\nu(L) := \bigcup_{x \in L} (T_x L)^{d\alpha}/T_x L.$$

It is naturally a symplectic vector bundle.

Remark 5.7. If a group G acts on a manifold M preserving a contact form α and L is an invariant, isotropic submanifold, then the symplectic normal bundle $\nu(L)$ has a natural G action which preserves the symplectic structure on $\nu(L)$.

In [We] Weinstein proved that the symplectic normal bundle completely determines the contact germ of the corresponding isotropic embedding. Theorem 4.1 is an equivariant version of Weinstein's result in the special case where the isotropic submanifold is a group orbit. We now proceed with the details of the proof of Theorem 4.1.

The proof is a slight modification of the standard argument in the symplectic case. We use the notation in the statement of the theorem. Let Y denote the Reeb vector field. Choose a G -invariant complex structure J on the symplectic vector bundle $\ker \alpha \rightarrow M$ compatible with the symplectic structure, that is, choose it so that $d\alpha(\cdot, J\cdot)|_{\ker \alpha}$ is a positive definite inner product. The inner product is G -invariant by construction. By declaring Y to be a unit vector field orthogonal to $\ker \alpha$ we can extend the inner product on $\ker \alpha$ to a G -invariant metric g on M .

Because the point x lies in the zero level set of the moment map, the orbit $G \cdot x$ is an isotropic submanifold of (M, α) . Since $T_x(G \cdot x)$ is isotropic in $(\ker \alpha_x, d\alpha_x)$, $T_x(G \cdot x)$ and $J(T_x(G \cdot x))$ are g -orthogonal by construction of g . Moreover $T_x(G \cdot x) \oplus J(T_x(G \cdot x))$ is a symplectic subspace of $(\ker \alpha_x, d\alpha_x)$, and the g -orthogonal complement V to $T_x(G \cdot x) \oplus J(T_x(G \cdot x))$ is also a symplectic subspace. The isotropy group H of x acts on $\ker \alpha_x$ preserving the decomposition $\ker \alpha_x = T_x(G \cdot x) \oplus J(T_x(G \cdot x)) \oplus V$ and the symplectic form $d\alpha_x|_{\ker \alpha_x}$. Note that V is isomorphic to $(T_x(G \cdot x))^{d\alpha}/T_x(G \cdot x)$. Consequently $G \times_H V \rightarrow G \cdot x = G/H$ is the symplectic normal bundle of the embedding $G \cdot x \hookrightarrow (M, \alpha)$ while $G \times_H (J(T_x(G \cdot x)) \times V \times \mathbb{R}Y_x)$ is the topological normal bundle of the embedding $G \cdot x \hookrightarrow M$. The map $v \mapsto d\alpha_x(v, \cdot)|_{T_x(G \cdot x)}$ identifies $J(T_x(G \cdot x))$ with $T_x^*(G \cdot x) \simeq (\mathfrak{g}/\mathfrak{h})^* \simeq \mathfrak{h}^\circ$, where as before \mathfrak{h}° denotes the annihilator of \mathfrak{h} in \mathfrak{g}^* . Therefore the topological normal bundle of the embedding of the orbit $G \cdot x$ in M is $G \times_H (\mathfrak{h}^\circ \times V \times \mathbb{R})$.

Denote the restriction of $d\alpha_x$ to V by ω_V . Let (W, β) denote the contactization of (V, ω_V) : $W = V \times \mathbb{R}$ and $\beta = \frac{1}{2}\iota(R)\omega_V + dt$ where R is the radial vector field on V and t is the variable along \mathbb{R} . Thus the normal bundle of the embedding $G \cdot x \hookrightarrow M$ can be written as $G \times_H (\mathfrak{h}^\circ \times W)$.

Remark that our various identifications are chosen in such a way that

$$\ker \alpha_x = T_x(G \cdot x) \oplus \mathfrak{h}^\circ \oplus V$$

and

$$d\alpha_x|_{\ker \alpha_x} = \omega + \omega_V,$$

where ω is the standard symplectic form on $T_x(G \cdot x) \oplus \mathfrak{h}^\circ = (\mathfrak{g}/\mathfrak{h}) \times (\mathfrak{g}/\mathfrak{h})^*$.

The exponential map defined by the metric g provides us with a G -equivariant diffeomorphism σ from a neighborhood U_0 of the zero section in $\mathcal{Y} = G \times_H (\mathfrak{h}^\circ \times W)$ to a neighborhood U of $G \cdot x$ in M such that $\sigma([1, 0, 0]) = x$ and $d\sigma_{[1, 0, 0]} : T_x(G \cdot x) \oplus \mathfrak{h}^\circ \oplus W \rightarrow T_x M$ is the identification above.

Therefore, by Darboux theorem (Theorem 5.2 above), in order to finish proving Theorem 4.1, it is enough to construct on \mathcal{Y} a G -invariant contact form ϵ with the following properties:

1. $\ker \epsilon_{[1, 0, 0]} = T_{[1, 0, 0]}(G/H) \oplus \mathfrak{h}^\circ \oplus V$ where we identified $T_{[1, 0, 0]}\mathcal{Y}$ with $T_{[1, 0, 0]}(G/H) \oplus \mathfrak{h}^\circ \oplus W = \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}^\circ \oplus W$ (we think of G/H as embedded in \mathcal{Y} as the zero section of $\mathcal{Y} \rightarrow G/H$).
2. $d\epsilon_{[1, 0, 0]}|_{\ker \epsilon_{[1, 0, 0]}} = \omega + \omega_V$
3. The moment map F for the action of G on (\mathcal{Y}, ϵ) is $F([g, \eta, w]) = Ad^\dagger(g)(\eta + i(\Phi_W(w)))$ where as before Φ_W is the moment map for the action of H on the contact vector space W .

The construction of ϵ is standard. The cotangent bundle T^*G is an exact symplectic manifold with a $G \times H$ -invariant symplectic potential, where G acts by the lift of the left multiplication and H by the lift of the multiplication on the right by the inverse. The group H also acts contactly on the contact vector space W . Hence the diagonal action of H on $T^*G \times W$ is contact. It is also free. It commutes with the trivial extension of the action of G on T^*G to $T^*G \times W$. Therefore the contact quotient $(T^*G \times W)//H$ is a contact manifold. Moreover the contact form on $(T^*G \times W)//H$ is G -invariant. It is routine to verify that a choice of an H -invariant splitting $\mathfrak{g}^* = \mathfrak{h}^\circ \oplus \mathfrak{h}^*$ allows one to identify the quotient $(T^*G \times W)//H$ with $G \times_H (\mathfrak{h}^\circ \times W) = \mathcal{Y}$ and the induced contact form ϵ on \mathcal{Y} has the desired properties. This finishes the proof of Theorem 4.1.

REFERENCES

- [Al] C. Albert, Le théorème de réduction de Marsden-Weinstein en géométrie cosymplectique et de contact, *J. Geom. Phys.* **6** (1989), no. 4, 627–649.
- [ACG] J. Arms, R. Cushman and M. Gotay, A universal reduction procedure for Hamiltonian group actions in **The geometry of Hamiltonian systems** (Berkeley, CA, 1989), 33–51, Math. Sci. Res. Inst. Publ., 22, Springer, New York, 1991.
- [AMM] J. Arms, J. Marsden and V. Moncrief, Symmetry and bifurcations of momentum mappings *Comm. Math. Phys.* **78** (1980/81), no. 4, 455–478.
- [BL] L. Bates and E. Lerman, Proper group actions and symplectic stratified spaces, *Pac. J. of Math.* (2) **181** (1997), 201–229.
- [Ge] H. Geiges, Constructions of contact manifolds, *Math. Proc. Cambridge Philos. Soc.* **121** (1997), no. 3, 455–464.
- [GM] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Berlin; New York: Springer-Verlag, 1988.
- [GS] V. Guillemin and S. Sternberg, Homogeneous quantization and multiplicities of group representations, *J. Funct. Anal.* **47** (1982), no. 3, 344–380.
- [K1] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Mathematical Notes **31**, Princeton University Press, Princeton, 1984.
- [K2] F. Kirwan, Convexity Properties of the Moment Mapping III, *Invent. Math.* **77** (1984), 547–552.
- [L] E. Lerman, Contact Cuts, *Israel J. Math.*, to appear. See <http://xxx.lanl.gov/abs/math.SG/0002041>.
- [O] J.-P. Ortega, thesis, University of California at Santa Cruz, 1998.
- [OR] J.-P. Ortega and T. Ratiu, in preparation.
- [P] R. Palais, On the existence of slices for actions of non-compact Lie groups, *Ann. Math.* **73** (1961), no. 2, 295–322.
- [SL] R. Sjamaar and E. Lerman, Stratified symplectic spaces and reduction, *Ann. of Math.* **134** (1991), 375–442.
- [We] A. Weinstein, Contact surgery and symplectic handlebodies, *Hokkaido Math. J.*, **20** (1991), no. 2, 241–251.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801

E-mail address: lerman@math.uiuc.edu

E-mail address: cwillett@math.uiuc.edu